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# Isolated singularities and positive solutions of elliptic equations in $\mathbb{R}^n$ (Potential Theory and Its Related Fields)

AUTHOR(S):

Murata, Minoru

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Isolated singularities and positive solutions of elliptic equations in  $R^n$

Tokyo Metropolitan University

村田 實

(Minoru Murata)

I would like to talk about isolated singularities of positive solutions of a second order linear elliptic equation

$$Pu \equiv - \sum_{j,k=1}^n \partial_j (a_{jk} \partial_k u) + \sum_{j=1}^n b_j \partial_j u + cu = 0 \quad (*)$$

in a domain of  $R^n$ . Here  $\partial_j = \partial/\partial x_j$ . The purpose of this talk is two-fold: (i) To give a relationship between positive solutions in a punctured ball and those in  $R^n$ ; (ii) to apply it to the study of isolated positive singularities of solutions.

1. Consider the equation (\*) in a domain  $D$  of  $R^n$ ,  $n \geq 2$ .

We assume that the coefficients are real-valued and satisfy the condition

$a_{jk} \in L_{\infty,loc}(D)$ ,  $b_j \in L_{2p,loc}(D)$  and  $c \in L_{p,loc}(D)$  for some  $p > n/2$ , and  $[a_{jk}(x)]_{j,k}$  is locally uniformly positive definite in  $D$ .

A positive solution of (\*) means a positive continuous function in the Sobolev space  $H_{loc}^1(D)$  of order 1 satisfying (\*) in the weak sense. Let  $\{D_j\}_{j=1}^\infty$  be a sequence of regular bounded domains which exhausts  $D$ . Let  $P_j$  be the Dirichlet realization of  $P$  on  $L_2(D_j)$ , and  $\sigma(P_j)$  the spectrum of  $P_j$ . Put

$$\Gamma(P,D) = \inf_{j=1,2,\dots} \inf \operatorname{Re} \sigma(P_j) .$$

Theorem (Allegretto-Piepenbrink-Agmon). There exists a positive solution of (\*) in  $D$  if and only if  $\Gamma(P,D) \geq 0$ .

Definition. We say that  $(P,D)$  is subcritical if there exists a positive Green's function in  $D$ , and that  $(P,D)$  is critical if  $\Gamma(P,D) \geq 0$  and there exists no positive Green's function in  $D$ .

Theorem. (i)  $(P,D)$  is subcritical if and only if there exists a function  $q$  in  $L_{p,loc}(D)$  such that  $q \geq 0$ ,  $q \neq 0$ , and  $\Gamma(P-q,D) \geq 0$ . (ii)  $(P,D)$  is critical if and only if  $\Gamma(P,D) \geq 0$  but there exists no function  $q$  as above.

We denote by  $H_+(P,D)$  the metric space of all positive solutions of (\*) in  $D$ , where the metric is the usual one generated by the maximum norms on  $D_j$ ,  $j=1,2,\dots$

Theorem. Suppose that  $(P, D)$  is critical. Then  $H_+(P, D) = \{Cu; C > 0\}$ , where  $u$  is a particular solution satisfying the integral equation

$$u(x) = \int G(x, y) q(y) u(y) dy$$

with  $q$  being a nonnegative continuous function with compact support which is not identically zero and  $G$  being the Green's function for  $P+q$  in  $D$ .

2. Now consider the equation (\*) in a punctured ball  $B^* = \{0 < |x| < R\}$ , where the coefficients satisfy the condition in Section 1 with  $D$  replaced by  $\{0 < |x| < R+1\}$ . Suppose that  $(P, B^*)$  is subcritical. Choose a positive continuous function  $g$  on  $\{0 < |x| \leq R\}$  satisfying

$$-\sum_{j,k} \partial_j (a_{jk} \partial_k g) + \sum_j b_j \partial_j g = 0 \text{ in } B^*.$$

Let  $H_+(P, B^*, \{0\}) = \{u \in H_+(P, B^*); u|_{|x|=R} = 0\}$ . For  $u$  in  $H_+(P, B^*, \{0\})$ , define a generalized Kelvin transform  $\kappa u$  by

$$(\kappa u)(y) = (u/g)(y/|y|^2).$$

Then  $\kappa u$  is a solution of  $P^1(\kappa u) = 0$  in  $B^{-1} \equiv \{y; |y| > R^{-1}\}$ , where  $P^1$  is an elliptic differential operator determined by  $P$  and  $g$ . We can show that

$P^1$  admits an extension  $\tilde{P}$  to  $R^n$  such that  $(\tilde{P}, R^n)$  is subcritical.

Theorem. There exists an isomorphism from  $H_+(P, B^*, \{0\})$  onto  $H_+(P^{\sim}, R^n)$  such that for any  $u$  in  $H_+(P, B^*, \{0\})$

$$(u/g)(y/|y|^2) \leq Tu(y) \leq C(u/g)(y/|y|^2) \quad \text{in } \{|y| > 2R^{-1}\},$$

where  $C$  is a positive constant independent of  $u$ . Furthermore,  $u$  is a minimal solution if and only if  $Tu$  is so.

Hint of proof. For  $v$  in  $H_+(P^{\sim}, R^n)$ , define  $\Pi v$  by

$$\Pi v = v - Bv,$$

where  $Bv$  is a solution of the boundary value problem:  $P^{\sim}(Bv) = 0$  in  $B^{-1}$ ,  $Bv = v$  on  $\{|x| = R^{-1}\}$ , and  $Bv$  is of minimal growth at infinity. Then  $\Pi$  is an isomorphism from  $H_+(P^{\sim}, R^n)$  onto  $H_+(P^{\sim}, B^{-1}, \{\infty\}) \equiv \{u \in H_+(P^{\sim}, B^{-1}); u|_{\partial B^{-1}} = 0\}$ . We put  $T = \Pi^{-1}_K$ .

Q.E.D.

We can also obtain analogous results starting with a subcritical operator in  $R^n$ .

**3.** I will give a few results concerning isolated positive singularities which I obtained by using the above theorem, although we could also show them directly.

Let  $P = -\Delta + V_0 + \lambda V_1$  in  $B^* = \{x \in R^2; 0 < |x| < R\}$ , where  $\lambda \in R^1$ ,  $V_1$  belongs to  $L_p(B^*)$  for some  $p \geq 2$  and satisfy  $\max(\pm V_1, 0) \neq 0$ , and

$$\begin{aligned} V_0(x) &= |x|^{\alpha_{2j}} \quad \text{for } x \text{ with } \theta_{2j-1} \leq \arg x \leq \theta_{2j}, \quad j=1, \dots, k, \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

where  $k$  is a natural number,  $0 = \theta_0 < \theta_1 < \dots < \theta_{2k} = 2\pi$ , and  $\alpha_{2j} < -2$  for  $j=1, \dots, k$ . We put  $\theta_i = \theta_i - \theta_{i-1}$  and  $\theta_{i-1/2} = \theta_{i-1} + \theta_i/2$ . Regarding  $\lambda$  as a parameter we have

Theorem. There exist  $a < 0$  and  $b > 0$  such that (i)  $(P, B^*)$  is subcritical if and only if  $a < \lambda < b$ ; and (ii)  $(P, B^*)$  is critical if and only if  $\lambda = a$  or  $b$ .

Theorem. If  $\lambda = a$  or  $b$ , then  $H_+(P, B^*, \{0\}) = \{Cp; C > 0\}$ , where the positive solution  $p(r, \phi)$ , with  $(r, \phi)$  being polar coordinates of  $R^2$ , has the following decay property as  $r \rightarrow 0$ :

$$\begin{aligned} p(r, \phi) &= o(r^{-\pi/\theta_{2j-1}}), & \theta_{2j-2} < \phi < \theta_{2j-1}, \\ &= O(\exp[-q_{2j}(\phi)r^{\alpha_{2j}/2 + 1}]), & \theta_{2j-1} < \phi < \theta_{2j}, \end{aligned}$$

where  $q_{2j}$  is a positive continuous function in  $(\theta_{2j-1}, \theta_{2j})$ .

Theorem. Suppose that  $a < \lambda < b$ . Then:

(i) The metric space

$$\begin{aligned} \text{Ex}H_+(P, B^*, \{0\}) &\equiv \{u \in H_+(P, B^*, \{0\}); u \text{ is extremal and} \\ &u(R/2, 0) = 1\} \end{aligned}$$

is homeomorphic to

$$\sigma = \bigcup_{i=1}^{2k} \sigma_i,$$

where  $\sigma_{2j} = \{\psi \in [0, 2\pi); |\psi - \theta_{2j-1/2}| \leq \theta_{2j}/2 + \pi/(\alpha_{2j} + 2)\}$  and  $\sigma_{2j-1} = \{\theta_{2j-3/2}\}$ .

(ii) The minimal solution  $P(r, \phi; \psi)$  corresponding to  $\psi$  in  $\sigma$  has the following asymptotics as  $r \rightarrow 0$ :

(a) For  $\psi = \theta_{2j-1}$ ,

$$P(r, \phi; \psi) = C(\psi) \chi_{2j-1}(\phi) r^{-\pi/\theta_{2j-1}} \sin[\pi(\phi - \theta_{2j-2})/\theta_{2j-1}] + p(r, \phi; \psi),$$

where  $C(\psi)$  is a positive constant,  $\chi_{2j-1}$  is the characteristic function of the set  $[\theta_{2j-2}, \theta_{2j-1}]$ , and  $p(r, \phi; \psi)$  has the same asymptotic property as in the above theorem.

(b) For  $\psi$  in  $\sigma_{2j}$ ,

$$P(r, \phi; \psi) = C(\psi) \chi_{2j}(\phi) \exp\left[-\frac{r^{\alpha_{2j}/2+1}}{\alpha_{2j}/2+1} \cos(\alpha_{2j}/2+1)(\phi-\psi)\right] [1 + o(1)] + p(r, \phi; \psi).$$

Of course, any positive solution is represented uniquely by integrating  $P(r, \phi; \psi)$  with respect to  $\psi$  by a positive Borel measure on  $\sigma$ .

An interesting open problem is: What is the Martin boundary over  $\{0\}$ ?

I conclude this talk with a theorem concerning the Dirichlet problem at  $\{0\}$ . Put

$$g(x) = \sum_{j=1}^k \{ \theta_{2j-1} P(x; \theta_{2j-3/2}) + \theta_{2j} \int_{\sigma_{2j}} P(x; \psi) \frac{d\psi}{|\sigma_{2j}|} \} + 1.$$

By convention, the second integral equals  $P(x; \theta_{2j})$  when  $\sigma_{2j} = \{\theta_{2j}\}$ , and is zero when  $\sigma_{2j} = \emptyset$ .

Theorem. For any continuous function  $f$  on  $\sigma$  there exists a unique solution  $u$  of the following problem

$$Pu = 0 \text{ in } B^*, \quad u|_{|x|=R} = 0, \quad u/g \in L_\infty(B^*),$$

$$\lim_{r \rightarrow 0} u(r, \phi)/g(r, \phi) = f(\phi) \text{ for any } \phi \in \sigma.$$

This theorem means, loosely speaking, the order of singularity is represented by that of  $g$ .

Finally, I should mention that there are several related results concerning the cardinal number of the set  $ExH_+$  of all normalized extremal positive solutions obtained by M. Nakai and his collaborators.



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